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Dimer Coverings on the Sierpinski Gasket

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Abstract We present the number of dimer coverings $N_d(n)$ on the Sierpinski gasket $SG_d(n)$ at stage n with dimension d equal to two, three, four or five. When the number of vertices, denoted as $v(n)$, of the Sierpinski gasket is an even number, $N_d(n)$ is the number of close-packed dimers. When the number of vertices is an odd number, no close-packed configurations are possible and we allow one of the outmost vertices uncovered. The entropy of absorption of diatomic molecules per site, defined as $S_{SG_d} = \lim_{n\to\infty} \ln N_d(n)/v(n)$, is calculated to be $\ln(2)/3$ exactly for SG_2. The numbers of dimers on the generalized Sierpinski gasket $SG_{d,b}(n)$ with $d = 2$ and $b = 3, 4, 5$ are also obtained exactly with entropies equal to $\ln(6)/7$, $\ln(28)/12$, $\ln(200)/18$, respectively. The number of dimer coverings for SG_3 is given by an exact product expression, such that its entropy is given by an exact summation expression. The upper and lower bounds for the entropy are derived in terms of the results at a certain stage for $SG_d(n)$ with $d = 3, 4, 5$. As the difference between these bounds converges quickly to zero as the calculated stage increases, the numerical value of S_{SG_d} with $d = 3, 4, 5$ can be evaluated with more than a hundred significant figures accurate.

Keywords Dimers · Sierpinski gasket · Entropy · Recursion relations · Exact solution

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1 Introduction

The enumeration of close-packed dimers $N(G)$ on a graph G was first considered by Fowler and Rushbrooke in enumerating the absorption of diatomic molecules on a surface [1]. The dimer coverings of a graph is a classical model in statistical physics and is called perfect matchings in mathematical literature. The dimer model on the square lattice was solved exactly by Kasteleyn [2] and Temperley and Fisher [3, 4]. The model is equivalent to various other statistical mechanical problems. For example, the zero-field partition function of Ising model on a planar lattice can be formulated as a dimer model on an associated planar lattice [5, 6]. It is also well known that there is a bijection between close-packed dimer coverings and spanning tree configurations on two related planar lattices [7]. A recent review on the enumeration of close-packed dimers on two-dimensional regular lattices is summarized in [8]. It is of interest to consider dimer coverings on self-similar fractal lattices which have scaling invariance rather than translational invariance. Fractals are geometric structures of generally noninteger Hausdorff dimension realized by repeated construction of an elementary shape on progressively smaller length scales [9, 10]. A well-known example of fractal is the Sierpinski gasket which has been extensively studied in several contexts [11–28]. Instead of using the method of Kasteleyn, it is more natural to use renormalization scheme [11, 12] to solve the dimer problem on the Sierpinski gasket which has finite ramification. We have succeeded in obtaining recursion relations for the more general number of dimer-monomers on the Sierpinski gasket [29], but the corresponding entropy does not assume a simple form. As we turn to the dimer model on the Sierpinski gasket, we observe that the recursion relations in [29] can be simplified, which allow us to solve entropies exactly for the two dimensional cases and calculate numbers of dimer coverings to higher dimension or side length reported below. A dimer coverings will leave at least one vertex uncovered when the total number of vertices is an odd number, e.g., the rectangular lattice with both length and width odd [30, 31]. The vacancies that are not covered by any dimers can be considered as occupied by monomers. Here when the number of vertices for a certain type of Sierpinski gasket is always an odd number, we shall allow a vacancy occurs on one of the outmost vertices. The purpose of this paper is to derive rigorously the numbers of dimer coverings on the two-dimensional Sierpinski gasket and its generalization, and obtain upper and lower bounds for the entropy on the Sierpinski gasket with dimension equal to three, four or five.

2 Preliminaries

We first recall some relevant definitions in this section. A connected graph (without loops) $G = (V, E)$ is defined by its vertex (site) and edge (bond) sets V and E [32, 33]. Let $v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in G. The degree or coordination number k_i of a vertex $v_i \in V$ is the number of edges attached to it. A k-regular graph is a graph with the property that each of its vertices has the same degree k. In general, one can associate a dimer (monomer) weight to each dimer (monomer) (see, for example [30]). For simplicity, all dimer (monomer) weights are set to one throughout this paper.

When the size of the graph increases as $v(G) \to \infty$, the number of dimer coverings $N(G)$ grows exponentially in $v(G)$. Here $N(G)$ is the number of close-packed dimers when $v(G)$ is an even number, and it is the number of almost close-packed dimers with a vacancy as mentioned above. The entropy of absorption of diatomic molecules per site is given by

$$S_G = \lim_{v(G)\to\infty} \frac{\ln N(G)}{v(G)}, \tag{2.1}$$

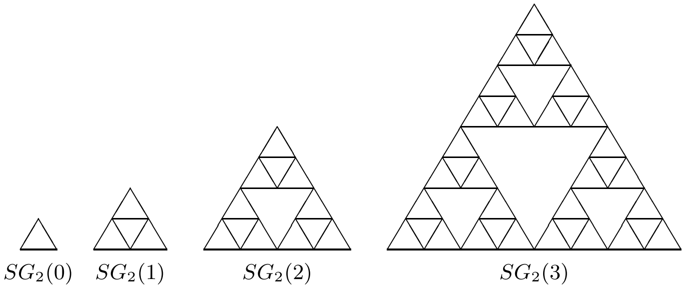
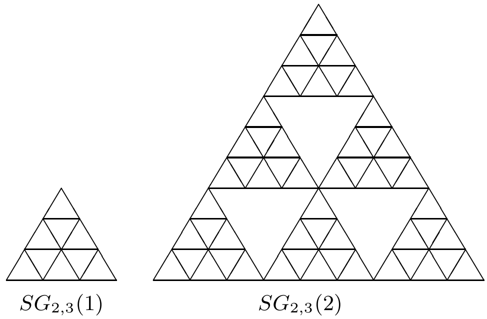


Fig. 1 The first four stages $n = 0, 1, 2, 3$ of the two-dimensional Sierpinski gasket $SG_2(n)$

Fig. 2 The generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$



where G, when used as a subscript in this manner, implicitly refers to the thermodynamic limit. Notice that we define the entropy per site rather than entropy per dimer. They differ by a factor of two in the thermodynamic limit.

The construction of the two-dimensional Sierpinski gasket $SG_2(n)$ at stage n is shown in Fig. 1. At stage $n = 0$, it is an equilateral triangle; while stage $n + 1$ is obtained by the juxtaposition of three n-stage structures. In general, the Sierpinski gaskets SG_d can be built in any Euclidean dimension d with fractal dimensionality $D = \ln(d + 1)/\ln 2$ [14]. For the Sierpinski gasket $SG_d(n)$, the numbers of edges and vertices are given by

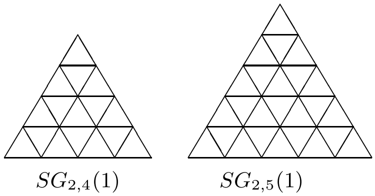
$$e(SG_d(n)) = \binom{d+1}{2}(d+1)^n = \frac{d}{2}(d+1)^{n+1}, \tag{2.2}$$

$$v(SG_d(n)) = \frac{d+1}{2}[(d+1)^n + 1]. \tag{2.3}$$

Except the $(d + 1)$ outmost vertices which have degree d, all other vertices of $SG_d(n)$ have degree $2d$. In the large n limit, SG_d is $2d$-regular.

The Sierpinski gasket can be generalized, denoted as $SG_{d,b}(n)$, by introducing the side length b which is an integer larger or equal to two [34]. The generalized Sierpinski gasket at stage $n + 1$ is constructed with b layers of stage n hypertetrahedrons. The two-dimensional $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$ are illustrated in Fig. 2, and those with $b = 4, 5$ at stage $n = 1$ in Fig. 3. The ordinary Sierpinski gasket $SG_d(n)$ corresponds to the $b = 2$ case, where the index b is neglected for simplicity. The Hausdorff dimension for $SG_{d,b}$ is given by $D = \ln\binom{b+d-1}{d}/\ln b$ [34]. For the two-dimensional Sierpinski gasket $SG_{2,b}(n)$ that will

Fig. 3 The generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 4, 5$ at stage $n = 1$



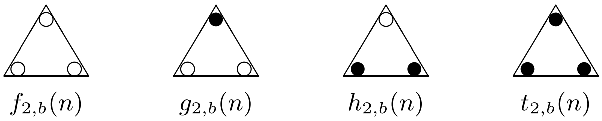


Fig. 4 Illustration for the configurations $f_{2,b}(n)$, $g_{2,b}(n)$, $h_{2,b}(n)$, and $t_{2,b}(n)$. Only the three outmost vertices are shown explicitly, where each *open circle* is vacant and each *solid circle* is occupied by a dimer

be considered here, the numbers of edges and vertices are given by

$$e(SG_{2,b}(n)) = 3\left[\frac{b(b+1)}{2}\right]^n, \tag{2.4}$$

$$v(SG_{2,b}(n)) = \frac{b+4}{b+2}\left[\frac{b(b+1)}{2}\right]^n + \frac{2(b+1)}{b+2}. \tag{2.5}$$

Notice that $SG_{d,b}$ is not k-regular even in the thermodynamic limit.

3 The Number of Dimer Coverings on $SG_{2,b}(n)$ with $b = 2, 3, 4, 5$

In this section we derive rigorously the numbers of dimer coverings on the two-dimensional Sierpinski gasket $SG_2(n)$, equivalently $SG_{2,2}(n)$, and the generalized $SG_{2,b}(n)$ with $b = 3, 4, 5$. Let us start with the definitions of the quantities to be used. They are illustrated in Fig. 4, where only the outmost vertices of $SG_{2,b}(n)$ are shown.

Definition 3.1 Consider the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ at stage n. (i) Define $f_{2,b}(n)$ as the number of dimer coverings such that the three outmost vertices are vacant. (ii) Define $g_{2,b}(n)$ as the numbers of dimer coverings such that one certain outmost vertex, say the topmost vertex as illustrated in Fig. 4, is occupied by a dimer while the other two outmost vertices are vacant. (iii) Define $h_{2,b}(n)$ as the numbers of dimer coverings such that one certain outmost vertex, say the topmost vertex as illustrated in Fig. 4, is vacant while the other two outmost vertices are occupied by dimers. (iv) Define $t_{2,b}(n)$ as the number of dimer coverings such that all three outmost vertices are occupied by dimers.

3.1 $SG_2(n)$

For the ordinary two-dimensional Sierpinski gasket, we use the notations $f_2(n)$, $g_2(n)$, $h_2(n)$, and $t_2(n)$ for simplicity. Because of rotational symmetry, there are three possible $g_2(n)$ and three possible $h_2(n)$ for non-negative integer n. The initial values at stage zero

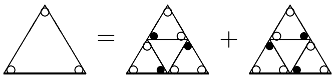


Fig. 5 Illustration for the expression of $f_2(2m+2)$

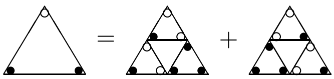


Fig. 6 Illustration for the expression of $h_2(2m+2)$

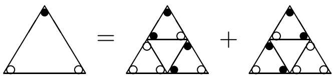


Fig. 7 Illustration for the expression of $g_2(2m+1)$

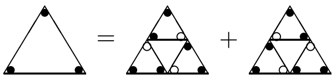


Fig. 8 Illustration for the expression of $t_2(2m+1)$

are $f_2(0)=1$, $g_2(0)=0$, $h_2(0)=1$, $t_2(0)=0$. The values at stage one are $f_2(1)=0$, $g_2(1)=2$, $h_2(1)=0$, $t_2(1)=2$. The value zero indicates that no such configurations are allowed. By (2.3), we have

$$v(SG_2(n)) = \frac{3}{2}(3^n+1) = 3^n + 2 + n + \sum_{j=2}^{n}\binom{n}{j}2^{j-1}, \tag{3.1}$$

where the Binomial expansion is used for $3^n = (2+1)^n$, such that the number of vertices for $SG_2(n)$ is odd for even n and even for odd n. Therefore, $f(n)$, $h(n)$ are always zero for odd n and $g(n)$, $t(n)$ are always zero for even n. Let us denote odd n as $2m+1$ and even n as $2m$ with non-negative integer m in the following discussion for $SG_2(n)$. These quantities satisfy simple recursion relations.

Lemma 3.1 *For any* $m \geq 0$,

$$f_2(2m+2) = 2g_2^3(2m+1), \tag{3.2}$$

$$h_2(2m+2) = 2g_2^2(2m+1)t_2(2m+1), \tag{3.3}$$

$$g_2(2m+1) = 2f_2(2m)h_2^2(2m), \tag{3.4}$$

$$t_2(2m+1) = 2h_2^3(2m). \tag{3.5}$$

Proof The Sierpinski gasket $SG_2(n+1)$ is composed of three $SG_2(n)$ with three pairs of vertices identified. For each pair of identified vertices, either one of them is originally occupied by a dimer while the other one is vacant. The number $f_2(2m+2)$ for $SG_2(2m+2)$ consists of two configurations where all three of the $SG_2(2m+1)$ are in the $g_2(2m+1)$ status as illustrated in Fig. 5, such that (3.2) is verified.

Similarly, $h_2(2m+2)$ and $g_2(2m+1)$, $t_2(2m+1)$ can be obtained with appropriate configurations of its three constituting blocks as illustrated in Figs. 6, 7 and 8 to verify (3.3), (3.4) and (3.5), respectively. □

It is elementary to solve $f_2(n)$, $g_2(n)$, $h_2(n)$, $t_2(n)$ in order to obtain the entropy for SG_2.

Theorem 3.1 *For the two-dimensional Sierpinski gasket $SG_2(n)$ at stage $n=2m$ or $n=2m+1$,*

$$\begin{cases} f_2(2m) = h_2(2m) = 2^{\gamma_2(2m)}, \\ f_2(2m+1) = h_2(2m+1) = 0, \end{cases} \tag{3.6}$$

$$\begin{cases} g_2(2m) = t_2(2m) = 0, \\ g_2(2m+1) = t_2(2m+1) = 2^{\gamma_2(2m+1)}, \end{cases} \tag{3.7}$$

where the exponent is

$$\gamma_2(n) = \frac{1}{2}(3^n - 1). \tag{3.8}$$

Define the number of dimer coverings $N(SG_2(n))$ in (2.1) equal to $h_2(n=2m)$ and equal to $t_2(n=2m+1)$ for even and odd n, respectively. With $v(SG_2(n)) = \frac{3}{2}(3^n+1)$, the entropy is given by

$$S_{SG_2} = \frac{1}{3}\ln 2 \simeq 0.23104906018\ldots. \tag{3.9}$$

It is intriguing that this entropy is the same as that for the Kagomé lattice [8, 35]. They share a common feature that they are both four-regular graphs in the thermodynamics limit (without worrying about boundary vertices). In passing, we notice that the above result is valid as one vacancy is allowed on a outmost vertex for even n with odd number of vertices. If one insists to always use the number of close-packed dimer coverings $t_2(n)$ in (2.1), the entropy does not exist.

3.2 $SG_{2,3}(n)$

For the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b=3$, we have

$$v(SG_{2,3}(n)) = \frac{7(6)^n+8}{5} = 6^n + 2 + 2\sum_{j=1}^{n}\binom{n}{j}5^{j-1} \tag{3.10}$$

by (2.5), such that the number of vertices is equal to three for $n=0$ and becomes even for all positive integer n. Therefore, $f_{2,3}(n)$ and $h_{2,3}(n)$ are always zero for positive integer n, while the initial values remain $f_{2,3}(0)=1$, $g_{2,3}(0)=0$, $h_{2,3}(0)=1$ and $t_{2,3}(0)=0$. The proof of the following recursion relations for $g_{2,3}(n)$ and $t_{2,3}(n)$ is given in the online archive version [36] of this paper but is omitted here to save space.

Lemma 3.2 *For any positive integer n,*

$$g_{2,3}(n+1) = 6g_{2,3}^5(n)t_{2,3}(n), \tag{3.11}$$

$$t_{2,3}(n+1) = 6g_{2,3}^4(n)t_{2,3}^2(n), \tag{3.12}$$

and for $n=0$,

$$g_{2,3}(1) = 6f_{2,3}^2(0)h_{2,3}^4(0) = 6, \tag{3.13}$$

$$t_{2,3}(1) = 6f_{2,3}(0)h_{2,3}^5(0) = 6. \tag{3.14}$$

It is elementary to solve $g_{2,3}(n)$ and $t_{2,3}(n)$ for positive n in order to obtain the entropy for $SG_{2,3}$.

Theorem 3.2 *For the generalized two-dimensional Sierpinski gasket $SG_{2,3}(n)$ at stage $n > 0$,*

$$g_{2,3}(n) = t_{2,3}(n) = 6^{\gamma_{2,3}(n)}, \tag{3.15}$$

where the exponent is

$$\gamma_{2,3}(n) = \frac{1}{5}(6^n - 1). \tag{3.16}$$

Define the number of dimer coverings $N(SG_{2,3}(n))$ in (2.1) equal to $t_{2,3}(n)$. With $v(SG_{2,3}(n)) = (7(6)^n + 8)/5$, the entropy is given by

$$S_{SG_{2,3}} = \frac{1}{7}\ln 6 \simeq 0.25596563846\ldots. \tag{3.17}$$

3.3 $SG_{2,4}(n)$

For the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 4$, we have

$$v(SG_{2,4}(n)) = \frac{4(10)^n + 5}{3} = 3 + \frac{4}{3}\sum_{j=1}^{n}\binom{n}{j}9^j \tag{3.18}$$

by (2.5), such that the number of vertices is always odd for any n. Therefore, $g_{2,4}(n)$ and $t_{2,4}(n)$ are zero for all n, while the initial values remain $f_{2,4}(0) = 1$, $g_{2,4}(0) = 0$, $h_{2,4}(0) = 1$ and $t_{2,4}(0) = 0$. The proof of the following recursion relations for $f_{2,4}(n)$ and $h_{2,4}(n)$ is given in the online archive version [36] of this paper but is omitted here to save space.

Lemma 3.3 *For any non-negative integer n,*

$$f_{2,4}(n+1) = 28 f_{2,4}^4(n) h_{2,4}^6(n), \tag{3.19}$$

$$h_{2,4}(n+1) = 28 f_{2,4}^3(n) h_{2,4}^7(n). \tag{3.20}$$

It is elementary to solve $f_{2,4}(n)$ and $h_{2,4}(n)$ in order to obtain the entropy for $SG_{2,4}$.

Theorem 3.3 *For the generalized two-dimensional Sierpinski gasket $SG_{2,4}(n)$ with non-negative integer n,*

$$f_{2,4}(n) = h_{2,4}(n) = 28^{\gamma_{2,4}(n)}, \tag{3.21}$$

where the exponent is

$$\gamma_{2,4}(n) = \frac{1}{9}(10^n - 1). \tag{3.22}$$

Define the number of dimer coverings $N(SG_{2,4}(n))$ in (2.1) equal to $h_{2,4}(n)$. With $v(SG_{2,4}(n)) = (4(10)^n + 5)/3$, the entropy is given by

$$S_{SG_{2,4}} = \frac{1}{12}\ln 28 \simeq 0.27768370918\ldots. \tag{3.23}$$

3.4 $SG_{2,5}(n)$

For the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b=5$, we have

$$v(SG_{2,5}(n)) = \frac{9(15)^n+12}{7} = 15^n + 2 + \frac{2}{7}\sum_{j=1}^{n}\binom{n}{j}14^j \tag{3.24}$$

by (2.5), such that the number of vertices is always odd for any n. Therefore, $g_{2,5}(n)$ and $t_{2,5}(n)$ are zero for all n, while the initial values remain $f_{2,5}(0)=1$, $g_{2,5}(0)=0$, $h_{2,5}(0)=1$ and $t_{2,5}(0)=0$. The figures of the recursion relations for $f_{2,5}(n)$ and $h_{2,5}(n)$ are too many to be shown here, and we state the following Lemma without proof.

Lemma 3.4 *For any non-negative integer n,*

$$f_{2,5}(n+1) = 200 f_{2,5}^6(n) h_{2,5}^9(n), \tag{3.25}$$

$$h_{2,5}(n+1) = 200 f_{2,5}^5(n) h_{2,5}^{10}(n). \tag{3.26}$$

It is elementary to solve $f_{2,5}(n)$ and $h_{2,5}(n)$ in order to obtain the entropy for $SG_{2,5}$.

Theorem 3.4 *For the generalized two-dimensional Sierpinski gasket $SG_{2,5}(n)$ with non-negative integer n,*

$$f_{2,5}(n) = h_{2,5}(n) = 200^{\gamma_{2,5}(n)}, \tag{3.27}$$

where the exponent is

$$\gamma_{2,5}(n) = \frac{1}{14}(15^n - 1). \tag{3.28}$$

Define the number of dimer coverings $N(SG_{2,5}(n))$ in (2.1) equal to $h_{2,5}(n)$. With $v(SG_{2,5}(n)) = (9(15)^n+12)/7$, the entropy is given by

$$S_{SG_{2,5}} = \frac{1}{18}\ln 200 \simeq 0.29435096480\ldots. \tag{3.29}$$

As the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ for any b is planar, it appears that the number of dimer coverings can be solved exactly. However, the number of configurations to be considered increases as b increases and the recursion relations must be derived individually for each b. We have been unable to obtain a general expression of the number of dimer coverings on $SG_{2,b}(n)$ for arbitrary b.

4 The Number of Dimer Coverings on $SG_d(n)$ with $d = 3, 4, 5$

In this section we present the number of dimer coverings on the Sierpinski gasket $SG_d(n)$ with $d=3,4,5$ which is not planar. Instead of solving exactly the entropies for these Sierpinski gaskets, we obtain accurate upper and lower bounds for them.

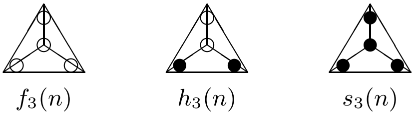


Fig. 9 Illustration for the dimer coverings $f_3(n)$, $h_3(n)$ and $s_3(n)$. Only the four outmost vertices are shown explicitly, where each *open circle* is vacant and each *solid circle* is occupied by a dimer

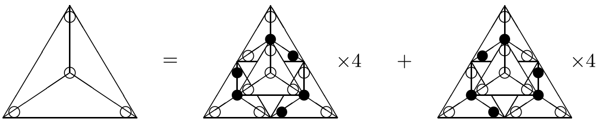


Fig. 10 Illustration for the expression of $f_3(n+1)$. The multiplication of four on the right-hand-side corresponds to the four possible orientations of $SG_3(n+1)$

4.1 $SG_3(n)$

For the three-dimensional Sierpinski gasket $SG_3(n)$, we use the following definitions.

Definition 4.1 Consider the three-dimensional Sierpinski gasket $SG_3(n)$ at stage n. (i) Define $f_3(n)$ as the number of dimer coverings such that the four outmost vertices are vacant. (ii) Define $h_3(n)$ as the number of dimer coverings such that two certain outmost vertices are occupied by dimers and the other two outmost vertices are vacant. (iii) Define $s_3(n)$ as the number of dimer coverings such that all four outmost vertices are occupied by dimers.

As the number of vertices for $SG_3(n)$ is always even by (2.3), we do not have the dimer coverings such that one certain outmost vertices is occupied by a dimer and the other three outmost vertices are vacant, or one certain outmost vertices is vacant and the other three outmost vertices are occupied by dimers. The quantities $f_3(n)$, $h_3(n)$, and $s_3(n)$ are illustrated in Fig. 9, where only the outmost vertices are shown. There are $\binom{4}{2} = 6$ equivalent $h_3(n)$. The initial values at stage zero are $f_3(0) = 1$, $h_3(0) = 1$, $s_3(0) = 3$. These quantities satisfy recursion relations.

Lemma 4.1 *For any non-negative integer n,*

$$f_3(n+1) = 8f_3(n)h_3^3(n), \tag{4.1}$$

$$h_3(n+1) = 4f_3(n)h_3^2(n)s_3(n) + 4h_3^4(n), \tag{4.2}$$

$$s_3(n+1) = 8h_3^3(n)s_3(n). \tag{4.3}$$

Proof The Sierpinski gasket $SG_3(n+1)$ is composed of four $SG_3(n)$ with six pairs of vertices identified. The number $f_3(n+1)$ for non-negative n consists of eight configurations where one of the $SG_3(n)$ are in the $f_3(n)$ status and the other three are in the $h_3(n)$ status as illustrated in Fig. 10, such that (4.1) is verified.

Similarly, $h_3(n+1)$ and $s_3(n+1)$ for $SG_3(n+1)$ can be obtained with appropriate configurations of its four constituting $SG_3(n)$ as illustrated in Figs. 11, and 12 to verify (4.2) and (4.3), respectively. □

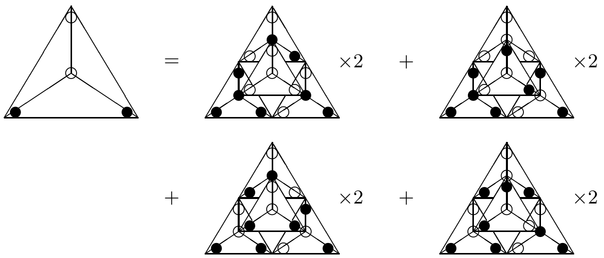


Fig. 11 Illustration for the expression of $h_3(n+1)$. The multiplication of two on the right-hand-side corresponds to the reflection symmetry with respect to the central vertical axis

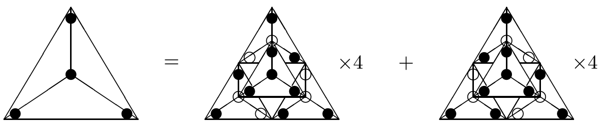


Fig. 12 Illustration for the expression of $s_3(n+1)$. The multiplication of four on the right-hand-side corresponds to the four possible orientations of $SG_3(n+1)$

The values of $f_3(n)$, $h_3(n)$, $s_3(n)$ for small n can be evaluated recursively by (4.1)–(4.3), but they grow exponentially, and do not have simple integer factorizations. To estimate the value of entropy for SG_3, we define the ratio

$$\alpha_3(n) = \frac{h_3(n)}{f_3(n)}, \tag{4.4}$$

and its limit

$$\alpha_3 \equiv \lim_{n\to\infty} \alpha_3(n). \tag{4.5}$$

Lemma 4.2 *Sequence* $\{\alpha_3(n)\}_{n=1}^{\infty}$ *decreases monotonically. The limit* α_3 *is equal to* $\sqrt{3}$.

Proof From (4.1) and (4.3), the ratio $s_3(n)/f_3(n)$ is invariant, that is equal to $s_3(0)/f_3(0) = 3$. Equation (4.2) can be modified to be

$$h_3(n+1) = 12 f_3^2(n) h_3^2(n) + 4h_3^4(n). \tag{4.6}$$

Although $\alpha_3(0) = 1$, it is clear that $\alpha_3(n)$ is bounded below by $\sqrt{3}$ for positive integer n because

$$h_3^2(n+1) - 3 f_3^2(n+1) = [12 f_3^2(n) h_3^2(n) - 4h_3^4(n)]^2 \ge 0 \tag{4.7}$$

for any $n \ge 0$. It follows that $\alpha_3(n)$ decreases for positive n because

$$\frac{h_3(n)}{f_3(n)} - \frac{h_3(n+1)}{f_3(n+1)} = \frac{h_3^2(n) - 3 f_3^2(n)}{2 f_3(n) h_3(n)} \ge 0, \tag{4.8}$$

which implies that the limit α_3 exists. From (4.1) and (4.6), we have

$$\frac{h_3(n+1)}{f_3(n+1)} = \frac{3}{2}\frac{f_3(n)}{h_3(n)} + \frac{1}{2}\frac{h_3(n)}{f_3(n)}. \tag{4.9}$$

By taking the large n limit in (4.9), α_3 is solved to be $\sqrt{3}$. □

The following general expressions for $f_3(n)$ and $h_3(n)$ can be established by induction. For a non-negative integer m and any positive integer $n > m$, we have

$$f_3(n) = 2^{\frac{2(4)^{n-m+1}-5-3(-1)^{n-m}}{10}} f_3(m)^{\frac{2(4)^{n-m}+3(-1)^{n-m}}{5}} \times h_3(m)^{\frac{3(4)^{n-m}-3(-1)^{n-m}}{5}} \prod_{j=2}^{n-m}[3+\alpha_3^2(n-j)]^{\frac{3(4)^{j-1}-3(-1)^{j-1}}{5}}, \tag{4.10}$$

$$h_3(n) = 2^{\frac{4^{n-m+1}-5+(-1)^{n-m}}{5}} f_3(m)^{\frac{2(4)^{n-m}-2(-1)^{n-m}}{5}} h_3(m)^{\frac{3(4)^{n-m}+2(-1)^{n-m}}{5}} \times \prod_{j=1}^{n-m}[3+\alpha_3^2(n-j)]^{\frac{3(4)^{j-1}+2(-1)^{j-1}}{5}}. \tag{4.11}$$

Here when $n - m = 1$, the product with lower limit two is defined to be one.

With above results, we have the following bounds for the entropy.

Lemma 4.3 *The entropy for the number of dimer coverings on $SG_3(n)$ is bounded*:

$$\frac{-\sqrt{3}\epsilon_3(m)^3}{720(4)^m} \le S_{SG_3} - \left\{\frac{2\ln f_3(m) + 3\ln h_3(m) + 5\ln 2 + \ln 3}{10(4)^m} + \frac{\sqrt{3}\epsilon_3(m)}{40(4)^m}\right\} \le \frac{\sqrt{3}\epsilon_3(m)^2}{40(4)^m[2\sqrt{3}-\epsilon_3(m)]}, \tag{4.12}$$

where $\epsilon_3(m)$ is defined as $\alpha_3(m) - \sqrt{3}$ and m is a positive integer.

Proof Substituting $N(G) = s_3(n) = 3f_3(n)$ in (2.1) for SG_3, we have

$$S_{SG_3} = \lim_{n\to\infty} \frac{\ln 3f_3(n)}{2(4^n+1)}, \tag{4.13}$$

where the factor of three in the logarithm can be neglected. By (4.10), we have

$$\ln f_3(n) = \frac{2(4)^{n-m}+3(-1)^{n-m}}{5}\ln f_3(m) + \frac{3(4)^{n-m}-3(-1)^{n-m}}{5}\ln h_3(m) + \frac{2(4)^{n-m+1}-5-3(-1)^{n-m}}{10}\ln 2 + \Delta_3(n,m), \tag{4.14}$$

where

$$\Delta_3(n,m) = \sum_{j=2}^{n-m} \frac{3(4)^{j-1}-3(-1)^{j-1}}{5}\ln[3+\alpha_3^2(n-j)], \tag{4.15}$$

which is bounded as follows.

By Lemma 4.2, we know $\epsilon_3(n)$ decreases monotonically to zero for positive integer n. $\epsilon_3(1) = \alpha_3(1) - \sqrt{3} = 2 - \sqrt{3}$. It is easy to find, by (4.9), that

$$\epsilon_3(n+1) = \frac{\epsilon_3(n)^2}{2(\sqrt{3}+\epsilon_3(n))}. \tag{4.16}$$

It is clear that if $X(m+1)=\frac{X(m)^2}{c}$ for non-negative integer m with $X(0)$ known and c constant, then

$$X(m)=\frac{X(0)^{2^m}}{c^{2^m-1}}. \tag{4.17}$$

As $\epsilon_3(n)$ in the denominator of (4.16) is close to zero when n is large, we have, for any integer $n \geq m$ with m fixed,

$$\epsilon_3(n+m)=\frac{\epsilon_3(m)^{2^n}}{(2\sqrt{3})^{2^n-1}}(1+o(n)), \tag{4.18}$$

where $o(n)\to 0$ as $n\to\infty$ and $o(n)$ is negative here. Replacing $\alpha_3(m)$ in (4.15) by $\sqrt{3}+\epsilon_3(m)$, $\Delta_3(n,m)$ can be rewritten as

$$\Delta_3(n,m)=\sum_{j=2}^{n-m}\frac{3(4)^{j-1}-3(-1)^{j-1}}{5}\ln[6+2\sqrt{3}\epsilon_3(n-j)+\epsilon_3(n-j)^2]. \tag{4.19}$$

Since $\epsilon_3(n)$ is small for positive n, the logarithmic term can be written as

$$\ln[6+2\sqrt{3}\epsilon_3(n-j)+\epsilon_3(n-j)^2]=\ln 6+\frac{\sqrt{3}}{3}\epsilon_3(n-j)\left[1-\frac{\xi_{n,j}\epsilon_3(n-j)^2}{18}\right], \tag{4.20}$$

where $\xi_{n,j}\in(0,1)$, so that

$$\begin{aligned}
&\Delta_3(n,m)\\
&\quad=\sum_{j=2}^{n-m}\frac{3(4)^{j-1}-3(-1)^{j-1}}{5}\left\{\ln 6+\frac{\sqrt{3}\epsilon_3(n-j)}{3}\left[1-\frac{\xi_{n,j}\epsilon_3(n-j)^2}{18}\right]\right\}\\
&\quad=\sum_{j=2}^{n-m}\frac{3\ln 6}{5}[(4)^{j-1}-(-1)^{j-1}]\\
&\qquad+\sum_{j=2}^{n-m}\frac{\sqrt{3}\epsilon_3(n-j)}{5}[(4)^{j-1}-(-1)^{j-1}]\left[1-\frac{\xi_{n,j}\epsilon_3(n-j)^2}{18}\right].
\end{aligned} \tag{4.21}$$

Because the $j=n-m$ term in the second summation gives the largest contribution among other terms, it is easy to see that

$$\frac{4^{n-m-1}\sqrt{3}\epsilon_3(m)(1-\frac{\epsilon_3(m)^2}{18})(1+o(n))}{5}\leq\Delta_3(n,m)-\frac{4^{n-m}\ln 6}{5}(1+o(n)). \tag{4.22}$$

On the other hand, $\sum_{j=2}^{n-m}\epsilon_3(n-j)4^{j-1}[1-\xi_{n,j}\epsilon_3(n-j)^2/18]$ in the second summation is less than $\sum_{i=0}^{n-m-2}4^{n-m-1}\epsilon_3(m+i)$. Using (4.18) and the inequality

$$\sum_{j=0}^{n-m-2}x^{2^j}=x+\sum_{j=1}^{n-m-2}x^{2^j}\leq x+\sum_{j=1}^{n-m-2}x^{2j}\leq\frac{x+x^2}{1-x^2}=\frac{x}{1-x} \tag{4.23}$$

for any $0 < x < 1$, we have

$$\Delta_3(n,m) - \frac{4^{n-m}\ln 6}{5}(1+o(n)) \le \frac{3(4)^{n-m}}{10}(1+o(n)) \sum_{i=0}^{n-m-2} \left(\frac{\epsilon_3(m)}{2\sqrt{3}}\right)^{2^i}$$
$$\le \frac{3(4)^{n-m}\epsilon_3(m)(1+o(n))}{10[2\sqrt{3}-\epsilon_3(m)]}. \tag{4.24}$$

The proof is completed by taking the infinite n limit in (4.13). □

The difference between the upper and lower bounds for S_{SG_3} quickly converges to zero as m increases, and we have the following proposition.

Proposition 4.1 *The entropy for the number of dimer coverings on the three-dimensional Sierpinski gasket $SG_3(n)$ in the large n limit is $S_{SG_3} = 0.42896389912\ldots$.*

By (4.18), we know

$$\epsilon_3(7) \le 2\sqrt{3}\left(\frac{2-\sqrt{3}}{2\sqrt{3}}\right)^{2^6}, \tag{4.25}$$

such that S_{SG_3} can be calculated with more than a hundred significant figures accurate when m is equal to seven in (4.12). It is too lengthy to be included here and is available from the authors on request. The above method to find upper and lower bounds for the entropy will be used for $SG_d(n)$ with $d = 4, 5$ in the following subsections.

Alternatively, $f_3(n)$ can be given as a product expression derived below.[1] Define the ratio

$$\bar{\alpha}_3(n) = \frac{h_3(n)}{\sqrt{3} f_3(n)}. \tag{4.26}$$

We have $\bar{\alpha}_3(0) = \sqrt{3}/3$, $\bar{\alpha}_3(1) = 2\sqrt{3}/3$, etc. By Lemma 4.2, $\bar{\alpha}_3(n)$ decreases to one from above for positive integer n. Equation (4.9) can be rewritten as

$$\bar{\alpha}_3(n+1) = \frac{1}{2}\left(\bar{\alpha}_3(n) + \frac{1}{\bar{\alpha}_3(n)}\right). \tag{4.27}$$

Changing the variable to

$$\beta_3(n) = \frac{\bar{\alpha}_3(n)-1}{\bar{\alpha}_3(n)+1}, \tag{4.28}$$

this equation becomes [37]

$$\beta_3(n+1) = \beta_3(n)^2, \tag{4.29}$$

which can be solved such that

$$\beta_3(n) = \beta_3(0)^{2^n}, \tag{4.30}$$

[1] The authors are indebted to D. Dhar for this derivation.

where $\beta_3(0) = \sqrt{3} - 2$. From (4.28), we obtain

$$\bar{\alpha}_3(n) = \frac{1 + \beta_3(n)}{1 - \beta_3(n)} = \frac{1 + (\sqrt{3} - 2)^{2^n}}{1 - (\sqrt{3} - 2)^{2^n}}. \tag{4.31}$$

Substituting $h_3(n) = \sqrt{3}\bar{\alpha}_3(n) f_3(n)$ into (4.1), $f_3(n)$ can be expressed in terms of $\bar{\alpha}_3(n)$:

$$f_3(n) = 8^{\frac{4^n-1}{3}} 3^{\frac{4^n-1}{2}} \prod_{j=0}^{n-1} \bar{\alpha}_3(j)^{4^{n-1-j}}, \tag{4.32}$$

such that the number of dimer coverings is given by

$$s_3(n) = 8^{\frac{4^n-1}{3}} 3^{\frac{4^n+1}{2}} \prod_{j=0}^{n-1} \bar{\alpha}_3(j)^{4^{n-1-j}}. \tag{4.33}$$

Finally, the entropy in (4.13) is given by

$$S_{SG_3} = \frac{1}{2}\ln 2 + \frac{1}{4}\ln 3 + \frac{3}{2}\sum_{j=0}^{\infty} \frac{\ln \bar{\alpha}_3(j)}{4^{j+1}}, \tag{4.34}$$

where $\ln \bar{\alpha}_3(j)$ approaches to zero for large j. We notice that if we estimate the value of S_{SG_3} with the summation in (4.34) evaluated up to j equal to a positive integer m, the deviation from the exact value is in the same order as the difference between the upper and lower bounds given in Lemma 4.3 using the same integer m.

4.2 $SG_4(n)$

For the four-dimensional Sierpinski gasket $SG_4(n)$, we use the following definitions.

Definition 4.2 Consider the four-dimensional Sierpinski gasket $SG_4(n)$ at stage n. (i) Define $f_4(n)$ as the number of dimer coverings such that the five outmost vertices are vacant. (ii) Define $h_4(n)$ as the number of dimer coverings such that two certain outmost vertices are occupied by dimers and the other three outmost vertices are vacant. (iii) Define $s_4(n)$ as the number of dimer coverings such that one certain outmost vertex is vacant and the other four outmost vertices are occupied by dimers.

By (2.3), we have

$$v(SG_4(n)) = \frac{5}{2}(5^n + 1) = 2(5)^n + 3 + \frac{1}{2}\sum_{j=1}^{n} \binom{n}{j} 4^j, \tag{4.35}$$

such that the number of vertices for $SG_4(n)$ is always odd. Therefore, we do not have the dimer coverings such that one certain outmost vertices is occupied by a dimer and the other four outmost vertices are vacant, or three certain outmost vertices are occupied by dimers and the other two outmost vertices are vacant, or all five outmost vertices are occupied by dimers. The quantities $f_4(n)$, $h_4(n)$, and $s_4(n)$ are illustrated in Fig. 13, where only the outmost vertices are shown. There are $\binom{5}{2} = 10$ equivalent $h_4(n)$ and $\binom{5}{1} = 5$ equivalent $s_4(n)$. The initial values at stage zero are again $f_4(0) = 1$, $h_4(0) = 1$, $s_4(0) = 3$.

We write a computer program to obtain following recursion relations.

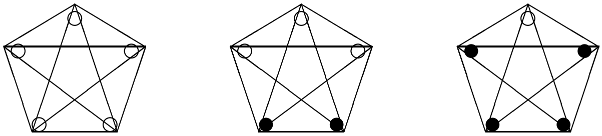


Fig. 13 Illustration for the dimer coverings $f_4(n)$, $h_4(n)$, $s_4(n)$. Only the five outmost vertices are shown explicitly, where each *open circle* is vacant and each *solid circle* is occupied by a dimer

Lemma 4.4 *For any non-negative integer n,*

$$f_4(n+1) = 40 f_4(n) h_4^3(n) s_4(n) + 24 h_4^5(n), \tag{4.36}$$

$$h_4(n+1) = 24 f_4(n) h_4^2(n) s_4^2(n) + 40 h_4^4(n) s_4(n), \tag{4.37}$$

$$s_4(n+1) = 8 f_4(n) h_4(n) s_4^3(n) + 56 h_4^3(n) s_4^2(n). \tag{4.38}$$

The values of $f_4(n)$, $h_4(n)$, $s_4(n)$ for small n can be evaluated recursively by (4.36)–(4.38), but they grow exponentially, and do not have simple integer factorizations. To estimate the value of entropy for SG_4, we define the ratios

$$\alpha_4(n) = \frac{h_4(n)}{f_4(n)}, \qquad \beta_4(n) = \frac{s_4(n)}{h_4(n)}, \tag{4.39}$$

and their limits

$$\alpha_4 \equiv \lim_{n\to\infty} \alpha_4(n), \qquad \beta_4 \equiv \lim_{n\to\infty} \beta_4(n). \tag{4.40}$$

Lemma 4.5 *Sequence $\{\alpha_4(n)\}_{n=1}^{\infty}$ decreases monotonically while sequence $\{\beta_4(n)\}_{n=1}^{\infty}$ increases monotonically. The ratio $\beta_4(n)/\alpha_4(n)$ for positive n increases monotonically to one.*

The proof of this Lemma is similar to that of Lemma 4.2, and is omitted here. It is available in the online archive version [36] of this paper. The numerical value of α_4 and β_4 is given by

$$\alpha_4 = \beta_4 = 0.850772150002\ldots \tag{4.41}$$

where more than a hundred significant figures can be evaluated when stage n in (4.39) is equal to seven.

The following general expressions for $h_4(n)$ and $s_4(n)$ can be established by induction. For a non-negative integer m and any positive integer $n > m$, we have

$$\begin{aligned} h_4(n) &= 2^{\frac{3(5)^{n-m}-3}{4}} h_4(m)^{\frac{3(5)^{n-m}+1}{4}} s_4(m)^{\frac{5^{n-m}-1}{4}} \\ &\quad \times \prod_{i=1}^{n-m} \left[5 + 3\frac{\beta_4(n-i)}{\alpha_4(n-i)}\right]^{\frac{3(5)^{i-1}+1}{4}} \prod_{j=2}^{n-m} \left[7 + \frac{\beta_4(n-j)}{\alpha_4(n-j)}\right]^{\frac{5^{j-1}-1}{4}}, \end{aligned} \tag{4.42}$$

$$\begin{aligned} s_4(n) &= 2^{\frac{3(5)^{n-m}-3}{4}} h_4(m)^{\frac{3(5)^{n-m}-3}{4}} s_4(m)^{\frac{5^{n-m}+3}{4}} \\ &\quad \times \prod_{i=2}^{n-m} \left[5 + 3\frac{\beta_4(n-i)}{\alpha_4(n-i)}\right]^{\frac{3(5)^{i-1}-3}{4}} \prod_{j=1}^{n-m} \left[7 + \frac{\beta_4(n-j)}{\alpha_4(n-j)}\right]^{\frac{5^{j-1}+3}{4}}. \end{aligned} \tag{4.43}$$

Here when $n - m = 1$, the products with lower limit two are defined to be one.

From above results, we have the following bounds for the entropy.

Lemma 4.6 *The entropy for the number of dimer coverings on $SG_4(n)$ is bounded*:

$$-\frac{7\epsilon_4(m)^2}{640(5)^m[1-\frac{\epsilon_4(m)}{16}]} \le S_{SG_4} - \left\{\frac{3\ln h_4(m)+\ln s_4(m)+6\ln 2}{10(5)^m} - \frac{\epsilon_4(m)}{40(5)^m}\right\} \le 0, \tag{4.44}$$

where m is a positive integer and $\epsilon_4(n)$ is defined as $1-\beta_4(n)/\alpha_4(n)$.

The proof of this Lemma is similar to that of Lemma 4.3, and is omitted here. It is available in the online archive version [36] of this paper. The difference between the upper and lower bounds for S_{SG_4} quickly converges to zero as m increases, and we have the following proposition.

Proposition 4.2 *The entropy for the number of dimer coverings on the four-dimensional Sierpinski gasket $SG_4(n)$ in the large n limit is $S_{SG_4} = 0.56337479920\ldots$.*

The numerical value of S_{SG_4} can be calculated with more than a hundred significant figures accurate when m in (4.44) is equal to six. It is too lengthy to be included here and is available from the authors on request.

4.3 $SG_5(n)$

For the five-dimensional Sierpinski gasket $SG_5(n)$, we use the following definitions.

Definition 4.3 Consider the five-dimensional Sierpinski gasket $SG_5(n)$ at stage n. (i) Define $f_5(n)$ as the number of dimer coverings such that the six outmost vertices are vacant. (ii) Define $g_5(n)$ as the number of dimer coverings such that one certain outmost vertex is occupied by a dimer and the other five outmost vertices are vacant. (iii) Define $h_5(n)$ as the number of dimer coverings such that two certain outmost vertices are occupied by dimers and the other four outmost vertices are vacant. (iv) Define $r_5(n)$ as the number of dimer coverings such that three certain outmost vertices are occupied by dimers and the other three outmost vertices are vacant. (v) Define $s_5(n)$ as the number of dimer coverings such that two certain outmost vertices are vacant and the other four outmost vertices are occupied by dimers. (vi) Define $t_5(n)$ as the number of dimer coverings such that one certain outmost vertex is vacant and the other five outmost vertices are occupied by dimers. (vii) Define $u_5(n)$ as the number of dimer coverings such that all six outmost vertices are occupied by dimers.

The quantities $f_5(n)$, $g_5(n)$, $h_5(n)$, $r_5(n)$, $s_5(n)$, $t_5(n)$ and $u_5(n)$ are illustrated in Fig. 14, where only the outmost vertices are shown. The initial values are $f_5(0)=1$, $g_5(0)=0$, $h_5(0)=1$, $r_5(0)=0$, $s_5(0)=3$, $t_5(0)=0$, $u_5(0)=15$. For the five-dimensional Sierpinski gasket $SG_5(n)$, the number of vertices is equal to six for $n=0$ and odd for all positive integer n by (2.3). Therefore, $f_5(n)$, $h_5(n)$, $s_5(n)$, $u_5(n)$ are always zero for positive integer n. There are $\binom{6}{1}=6$ equivalent $g_5(n)$ and $t_5(n)$, and $\binom{6}{3}=20$ equivalent $r_5(n)$.

We write a computer program to obtain following recursion relations.

Lemma 4.7 *For any positive integer n,*

$$g_5(n+1) = 40g_5^3(n)r_5(n)t_5^2(n) + 560g_5^2(n)r_5^3(n)t_5(n) + 424g_5(n)r_5^5(n), \tag{4.45}$$

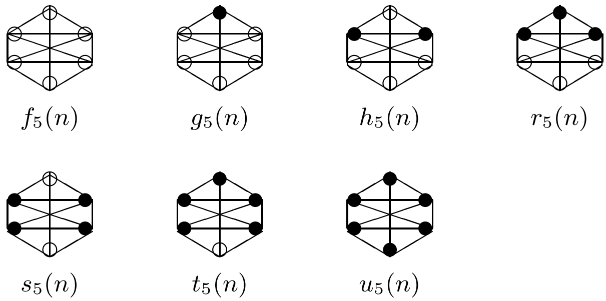


Fig. 14 Illustration for the dimer coverings $f_5(n)$, $g_5(n)$, $h_5(n)$, $r_5(n)$, $s_5(n)$, $t_5(n)$, $u_5(n)$. Only the six outmost vertices are shown explicitly, where each *open circle* is vacant and each *solid circle* is occupied by a dimer

$$r_5(n+1) = 4g_5^3(n)t_5^3(n) + 252g_5^2(n)r_5^2(n)t_5^2(n) + 636g_5(n)r_5^4(n)t_5(n) + 132r_5^6(n), \tag{4.46}$$

$$t_5(n+1) = 40g_5^2(n)r_5(n)t_5^3(n) + 560g_5(n)r_5^3(n)t_5^2(n) + 424r_5^5(n)t_5(n), \tag{4.47}$$

and for $n = 0$,

$$g_5(1) = 280f_5(0)h_5^2(0)s_5^3(0) + 40f_5(0)h_5^3(0)s_5(0)u_5(0) + 680h_5^4(0)s_5^2(0) + 24h_5^5(0)u_5(0) = 15840, \tag{4.48}$$

$$r_5(1) = 72f_5(0)h_5^2(0)s_5^2(0)u_5(0) + 120f_5(0)h_5(0)s_5^4(0) + 712h_5^3(0)s_5^3(0) + 120h_5^4(0)s_5(0)u_5(0) = 44064, \tag{4.49}$$

$$t_5(1) = 40f_5(0)h_5(0)s_5^3(0)u_5(0) + 280h_5^3(0)s_5^2(0)u_5(0) + 24f_5(0)s_5^5(0) + 680h_5^2(0)s_5^4(0) = 114912. \tag{4.50}$$

The values of $g_4(n)$, $r_4(n)$, $t_4(n)$ for small positive n can be evaluated recursively by (4.45)–(4.47), but they grow exponentially, and do not have simple integer factorizations. To estimate the value of entropy for SG_5, we define the ratio

$$\alpha_5(n) = \frac{r_5(n)}{g_5(n)}, \tag{4.51}$$

and its limit

$$\alpha_5 \equiv \lim_{n\to\infty} \alpha_5(n). \tag{4.52}$$

From (4.45) and (4.47), the ratio $t_5(n)/g_5(n)$ is invariant. Defined the ratio as c, then

$$c = \frac{t_5(1)}{g_5(1)} = \frac{399}{55}. \tag{4.53}$$

Equations (4.45) and (4.46) can be modified to be

$$g_5(n+1) = 8g_5^5(n)r_5(n)P_5(n), \tag{4.54}$$

$$r_5(n+1) = 4g_5^6(n)Q_5(n), \tag{4.55}$$

where

$$P_5(n) = 5c^2 + 70c\alpha_5^2(n) + 53\alpha_5^4(n), \tag{4.56}$$

$$Q_5(n) = c^3 + 63c^2\alpha_5^2(n) + 159c\alpha_5^4(n) + 33\alpha_5^6(n). \tag{4.57}$$

Lemma 4.8 *Sequence $\{\alpha_5(n)\}_{n=1}^{\infty}$ decreases monotonically. The limit α_5 is equal to $\sqrt{399/55}$.*

The proof of this Lemma is similar to that of Lemma 4.2, and is omitted here. It is available in the online archive version [36] of this paper.

The following general expressions for $g_5(n)$ and $r_5(n)$ can be established by induction. For a non-negative integer m and any positive integer $n > m$, we have

$$g_5(n) = 2^{\frac{8(6)^{n-m}-7-(-1)^{n-m}}{14}} g_5(m)^{\frac{6^{n-m+1}+(-1)^{n-m}}{7}} r_5(m)^{\frac{6^{n-m}-(-1)^{n-m}}{7}} \times \prod_{i=1}^{n-m} P_5(n-i)^{\frac{6^i-(-1)^i}{7}} \prod_{j=2}^{n-m} Q_5(n-j)^{\frac{6^{j-1}+(-1)^j}{7}}, \tag{4.58}$$

$$r_5(n) = 2^{\frac{4(6)^{n-m}-7+3(-1)^{n-m}}{7}} g_5(m)^{\frac{6^{n-m+1}-6(-1)^{n-m}}{7}} r_5(m)^{\frac{6^{n-m}+6(-1)^{n-m}}{7}} \times \prod_{i=2}^{n-m} P_5(n-i)^{\frac{6^i+6(-1)^i}{7}} \prod_{j=1}^{n-m} Q_5(n-j)^{\frac{6^{j-1}-6(-1)^j}{7}}. \tag{4.59}$$

Here when $n-m=1$, the products with lower limit two are defined to be one.

From above results, we have the following bounds for the entropy.

Lemma 4.9 *The entropy for the number of dimer coverings on $SG_5(n)$ is bounded*:

$$0 \le S_{SG_5} - \left\{ \frac{2\ln g_5(m)}{7(6)^m} + \frac{\ln r_5(m)}{21(6)^m} + \frac{14\ln 2}{21(6)^m} + \frac{\ln c}{7(6)^m} + \frac{9\epsilon_5(m)}{56\sqrt{c}(6)^m} \right\} \le \frac{279\epsilon_5(m)^2}{448c(6)^m[1-\frac{\epsilon_5(m)}{8\sqrt{c}}]}, \tag{4.60}$$

where m is a positive integer and $\epsilon_5(n)$ is defined as $\alpha_5(n) - \sqrt{c}$.

The proof of this Lemma is similar to that of Lemma 4.3, and is omitted here. It is available in the online archive version [36] of this paper. The difference between the upper and lower bounds for S_{SG_5} quickly converges to zero as m increases, and we have the following proposition.

Proposition 4.3 *The entropy for the number of dimer coverings on the five-dimensional Sierpinski gasket $SG_5(n)$ in the large n limit is $S_{SG_5} = 0.67042810305\ldots$.*

The numerical value of S_{SG_5} can be calculated with more than a hundred significant figures accurate when m in (4.60) is equal to six. It is too lengthy to be included here and is available from the authors on request.

We notice that the convergence of the upper and lower bounds of the entropy for dimer coverings on $SG_d(n)$ is about the same for $d = 3, 4, 5$, similar to the results observed in [29] for the dimer-monomer model on $SG_d(n)$.

Table 1 Numerical values of $S_{SG_{d,b}}$, $S_{\mathcal{L}_d}$ and the ratios $S_{SG_{d,b}}/z_{SG_{d,b}}$, $S_{SG_d}/S_{\mathcal{L}_d}$. The last digits given are rounded off

d	b	D	$S_{SG_{d,b}}$	$S_{SG_{d,b}}/z_{SG_{d,b}}$	$S_{\mathcal{L}_d}$	$S_{SG_d}/S_{\mathcal{L}_d}$
2	2	1.585	$\frac{1}{3}\ln 2 \simeq 0.2310490602$	0.3520510271	$G/\pi \simeq 0.2915609040$	0.7924555624
2	3	1.631	$\frac{1}{7}\ln 6 \simeq 0.2559656385$	0.3811183712	–	–
2	4	1.661	$\frac{1}{12}\ln 28 \simeq 0.2776837092$	0.4054532859	–	–
2	5	1.683	$\frac{1}{18}\ln 200 \simeq 0.2943509648$	–	–	–
3	2	2	0.4289638991	0.5491430497	0.4465	0.9608
4	2	2.322	0.5633747992	0.6425502211	–	–
5	2	2.585	0.6704281031	–	–	–

5 Summary

Compare the present results with those in [29], it is clear that the number of dimer coverings on the Sierpinski gasket $SG_d(n)$ is less than that of dimer-monomers. The asymptotic growth constant $z_{SG_{d,b}}$ for the dimer-monomer model defined as (2.1) of [29] corresponds to the entropy $S_{SG_{d,b}}$ for the dimer coverings defined in (2.1). We summarize the values of $S_{SG_{d,b}}$ and the ratio $S_{SG_{d,b}}/z_{SG_{d,b}}$ in Table 1. The value of S_{SG_d} increases as dimension d increases. Similarly for the generalized two-dimensional Sierpinski gasket, the exact value of $S_{SG_{2,b}}$ increases slightly as b increases. For the cases studied, the ratio S_{SG_d}/z_{SG_d} also increases as dimension d increases, and $S_{SG_{2,b}}/z_{SG_{2,b}}$ increases slightly as b increases.

It is interesting to compare entropy of dimer coverings on the Sierpinski gasket SG_d with that on the d-dimensional hypercubic lattice $\mathcal{L}_d$ which is also $2d$-regular. The entropy of the square lattice was known to be G/π [4], where G is the Catalan number, for decades, while the entropy of the simple cubic lattice was estimated to be 0.44647 [38]. They are relatively larger than the entropies on SG_d with $d = 2, 3$ presented here. The values of $S_{\mathcal{L}_d}$ and the ratio $S_{SG_d}/S_{\mathcal{L}_d}$ for $d = 2, 3$ are given in Table 1. It appears that as the d increases, the value S_{SG_d} approaches to the value $S_{\mathcal{L}_d}$ from below. As we have obtained the highly accurate value for the entropy on SG_d with $d = 4, 5$, there is no numerical estimation for the entropy on $\mathcal{L}_d$ with $d \geq 4$, to the best of our knowledge.

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